Problem Set 7
Due **Thursday 5/15/2025**

Note: You are not allowed to use AI assistance for generating solutions or code for these problem sets.

If you are using a python notebook, please submit your python notebook ipynb file with your PSet. Otherwise you must submit your code file in a way that it can be run. Please also submit proof that your code runs properly and yields the desired answers. Finally, please clearly mark what sections of your code are for what problems.

Not following these instructions will result in points deducted for this and all future PSets.

Problem 7-1 [16 points] The Cooper pair box. Here we will study numerically the behavior of the Cooper pair box (CBP) in three different regimes (charge qubit - quantronium - transmon). In the charge basis, the Hamiltonian is given by:

$$H = 4E_C(n - n_g)^2 - \frac{E_J}{2}\sum_n |n+1\rangle\langle n| + |n\rangle\langle n+1|, \tag{1}$$

with $E_C = e^2/(2C)$ the single electron charging energy, E_J the Josephson energy, and $n_g = CV_g/(2e)$ the gate voltage expressed in units of Cooper pairs. Also n is the number operator for Cooper pairs on C; therefore n_g should be considered multiplied by the identity operator I. The sum over n runs over all charges, so we'll have to make an approximation by only considering a few Cooper pair charge states, between $\pm N_c$. You can experiment to see how few you can get away with and still get a good result. Also the Joule is not a natural experimental unit, so divide by h and express all energies in this Pset in terms of GHz.

(a) In QuTiP, construct the $n = n|n\rangle\langle n|$ operator (don't you love this notation?). This should be a diagonal matrix with the n's on the diagonal, from $-N_c$ to $+N_c$.

(b) Now construct the full electrostatic energy $4E_C(n - n_g)^2$ (note again n_g is a number so here we should write $n_g I$).

(c) Construct the Josephson coupling operator (the second term in the Hamiltonian), which will consist of couplings between adjacent charge states.

(d) Add the results of (b) and (c) together to get the full Hamiltonian. Find the eigenvalues and eigenvectors of this Hamiltonian as a function of E_J, E_c,and n_g. Plot the first 5 energy levels as a function of n_g for the following parameters:

   (i) $E_J = 5$ GHz, $E_C = 20$ GHz (charge qubit regime)

   (ii) $E_J = 5$ GHz, $E_C = 5$ GHz (quantronium regime)

   (iii) $E_J = 50$ GHz, $E_C = 0.5$ GHz (transmon regime)

   Take care to plot everything on the same energy scale so it is easy to compare.

(e) Plot the transition frequency E_{01} as a function of n_g. Notice that the maxima are at n_g integer and minima at n_g half-integer.

(f) Plot the anharmonicity $\eta = E_{12} - E_{01}$ as a function of n_g.

(g) Our drive, measurement, and gate speeds will be proportional to the transition dipole matrix element. If we couple via charge (which is typical), the relevant matrix element is $\langle 0|n|1\rangle$. To calculate this, take the eigenvectors you found for the ground and first excited states in (d) and find the inner product with the n operator you made in (a). Plot this matrix element for the three cases in part (d).

(h) Now let's be more quantitative about how things change as a function of the ratio E_J/E_C. We will keep the "plasma" energy $E_P = \sqrt{8E_C E_J} = 5$ GHz fixed. Plot E_{01}, $\langle 0|n|1\rangle$, and η vs. E_J/E_C. To simplify the plots and calculations, don't do this for all n_g, just at $n_g \sim 0$. However for numerical stability it's best to move off zero by a small amount (0.001 or so). Also plot the band dispersion $\epsilon_{01} = |E_{01}(n_g = 0) - E_{01}(n_g = 1/2)|$. This gives the maximum change of the qubit frequency due to charge noise.

Problem 7-2 [15 points] Let's prepare a cat state! Assume we now have a microwave cavity. To have fun, we want to introduce a Kerr nonlinearity for the cavity (to achieve that, you can couple your cavity to a nonlinear device like a transmon qubit and inherit nonlinearity from there). In rotating frame, we can then model the cavity with the Hamiltonian

$$H = \frac{\chi}{2}\hat{n}^2 = \frac{\chi}{2}(a^\dagger a)^2. \tag{2}$$

To initialize the cavity, we apply a linear drive to get a coherent state $|\alpha\rangle$ which, under the Fock basis, can be written as

$$|\alpha\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle. \tag{3}$$

Let your state evolve under H. At time t, your state can be written

$$|\psi(t)\rangle = e^{-i\hat{H}t}|\alpha\rangle. \tag{4}$$

(a) Can you write down $|\psi(t)\rangle$ explicitly in the Fock basis $\{|n\rangle\}$?

(b) After a time $t = \pi/\chi$, your state $|\psi(t = \pi/\chi)\rangle$ will be a linear superposition of two coherent states, say

$$|\psi(t = \pi/\chi)\rangle = \frac{1}{\sqrt{\mathcal{N}}}(e^{i\phi_1}|\beta_1\rangle + e^{i\phi_2}|\beta_2\rangle), \tag{5}$$

where $\mathcal{N}$ is a normalization factor. This type of state is called a "cat state". Can you determine what β_1, β_2 and the corresponding ϕ_1, ϕ_2 are? (Hint: treat the phase gained from e^{-iHt} for odd $|n\rangle$ and even $|n\rangle$ separately).

(c) Can you plot the Wigner function of $|\psi(t = \pi/\chi)\rangle$ in QuTiP? Choose $\alpha = 3$. To represent the state $|\psi(t)\rangle$ in QuTiP, you need to choose a proper cutoff dimension. You will find some reference code in the discussion section materials.

(d) Use QuTiP to calculate $|\psi(t = \pi/2\chi)\rangle$ and plot its Wigner function.

(e) After a revival time $t = T_{\text{rev}}$, your $|\psi(T_{\text{rev}})\rangle$ will again be a coherent state $|\beta\rangle$. Determine the minimal T_{rev} that leads to such a coherent state, and what the corresponding β is.