Problem Set 5

Due **Tuesday 4/29/2025**

Note: You are not allowed to use AI assistance for generating solutions or code for these problem sets.

If you are using a python notebook, please submit your python notebook ipynb file with your PSet. Please also submit proof that your code runs properly and yields the desired answers. Finally, please clearly mark what sections of your code are for what problems.

Not following these instructions will result in points deducted for this and all future PSets.

Problem 5-1 [21 points] Optimal control of an NV-center. Consider an NV center with two carbon-13 nuclei nearby, with energy levels of the electron spin system identical to that shown in Extended Data Figure 6 of Waldherr et al., "Quantum error correction in a solid-state hybrid spin register," *Nature* **506**, 204-207 (2014). You only have to consider the four levels arising from the two carbon atoms (you can disregard the nitrogen nuclear spin). You will use QuTiP to implement a simplified implementation of quantum optimal control, similar to what was done in the paper.

(a) To begin, use QuTiP to implement the natural CC-2π gate, where you drive resonantly on the $|011\rangle \rightarrow |111\rangle$ ($|NV, C_1, C_2\rangle$) transition, assuming a Rabi frequency of 0.1 MHz and the energy splittings from the paper.

(b) Next, assume that you drive the system with a frequency fixed at the center of each of the four transition frequencies, with complete control of the amplitude and phase of the applied microwave field for arbitrary slices of time. Formulate the Hamiltonian in the optimal control framework as $H_{total} = H_{drift} + \sum_j H_{j,control}$.

(c) Use quantum optimal control to optimize for a unitary that is closest to the desired CC-2π gate, using the same total timescale as the resonant gate implemented in part (a).

(d) Compare the errors (infidelities) between the resonant pulse and the optimized off-resonant gate with respect to the ideal CC-2π gate, using the same unitary matrix fidelity metric as QuTiP's `optimize_pulse_unitary()` command:
$\epsilon = 1 - \frac{1}{d}|Tr(U^\dagger_{target} U_{actual})|$

(e) Apply the resonant pulse as well as the optimized off-resonant pulse to each of the four ground basis states as well as their equal superposition: $|000\rangle$, $|001\rangle$, $|010\rangle$, $|011\rangle$, and $\frac{1}{2}(|000\rangle + |001\rangle + |010\rangle + |011\rangle)$. Compare the fidelities of the output states with respect to the output states of the ideal CC-2π gate using the fidelity() command.

(f) Comment on your results. Which gate is better? Is it better in all cases?

(g) BONUS (3 pts): Use quantum optimal control along with your understanding of the system to design an even better CC-2π gate in the same total time by, for example, adjusting the drive frequency, incorporating multiple drive frequencies, etc. Demonstrate that your gate is in fact better using the error metrics in previous parts.

Problem 5-2 [25 points + 2 bonus] Robust Rydberg CZ gate
Don't be afraid! You are still dealing with a 2-level system. In this problem, we will try to go through some very recent proposals to achieve robust CZ gate in neutral atoms, with the help of Rydberg interactions. These try to use composite pulse techniques to minimize quasi-static errors, including amplitude and detuning errors. The paper is Fromonteil et al. "Protocols for Rydberg entangling gates featuring robustness against quasi-static errors," *PRX Quantum* **4**, 020335 (2023).

(a) To do this problem, you need to use three levels $\{|0\rangle, |1\rangle, |r\rangle\}$ to model each atom, where $|0\rangle$ and $|1\rangle$ form the computational basis while $|r\rangle$ is the Rydberg level. You can use a global laser field to drive two atoms between $|1\rangle$ and $|r\rangle$ on-resonance together (since local addressing is a little difficult). The Hamiltonian in the rotating frame is given by Eq.(1) in the paper. In the strong blockade limit $V \to +\infty$, you can consider the Hamiltonian confined to eight dimensions, $H_B = P_B H P_B$, where $P_B = I \otimes I - |r,r\rangle\langle r,r|$ is the projection operator. Write down the matrix representation of H_B in the basis $\{|00\rangle, |01\rangle, |0r\rangle, |10\rangle, |r0\rangle, |11\rangle, |W\rangle, |A\rangle\}$ with $|W\rangle = \frac{1}{\sqrt{2}}(|1r\rangle + |r1\rangle)$ and $|A\rangle = \frac{1}{\sqrt{2}}(|1r\rangle - |r1\rangle)$. Briefly explain why it is sufficient to use knowledge from two-level systems to calculate the dynamics of this system.

(b) Next you want to build up entanglement following Fig. 1 (Eq.(2) in paper). Suppose you initialize your state in $|\psi_i\rangle = |+X\rangle|+X\rangle$ where $|+X\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. The unitary including amplitude errors will be

$$U(\epsilon) = U_x[\frac{(1+\epsilon)\pi}{2\sqrt{2}}]U_y[(1+\epsilon)\pi]U_x[\frac{(1+\epsilon)\pi}{\sqrt{2}}]U_y[(1+\epsilon)\pi]U_x[\frac{(1+\epsilon)\pi}{2\sqrt{2}}] \tag{1}$$

For the ideal case $\epsilon = 0$, what is the final state $|\psi_f(0)\rangle$? Verify you can get a Bell state from $|\psi_f(0)\rangle$ with only single-qubit operations.

(c) BONUS (2 pt) Can you reproduce Fig. 1?

(d) For the noisy case, can you plot the state preparation fidelity $F(\epsilon) = |\langle\psi_f(0)|\psi_f(\epsilon)\rangle|^2$ as ϵ varies? In the $\epsilon \ll 1$ regime, we have $(1-F) \propto \epsilon^n$; can you determine what n is?

(e) The authors claimed that their gate has "conditional robustness", that is, that you can improve the fidelity by performing measurement and keep the part which lies in the computational subspace $\{|0\rangle, |1\rangle\}^{\otimes 2}$. This means that you have

$$\rho_C(\epsilon) = \frac{P_C|\psi_f(\epsilon)\rangle\langle\psi_f(\epsilon)|P_C}{\langle\psi_f(\epsilon)|P_C|\psi_f(\epsilon)\rangle} \tag{2}$$

where $P_C = |00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 10| + |11\rangle\langle 11|$? Can you plot $F_C(\epsilon) = \langle\psi_f(0)|\rho_C(\epsilon)|\psi_f(0)\rangle$ as ϵ changes? In $\epsilon \ll 1$ limit, $(1-F_C) \propto \epsilon^{n'}$, can you find n'?

(f) In Sec. IV.A.2 the authors propose another sequence to achieve a "fully robust protocol," by applying two controlled-$(\frac{\pi}{2})$ gates sequentially (Fig. 2). First intuitively explain why the generalization shown in Fig. 2 gives controlled-$(\frac{\pi}{2})$ gates and can be further generalized to a controlled-(θ) gate with arbitrary θ. Then repeat what you did in part (d) with the "fully robust" CZ gate suffering from amplitude errors ϵ. Specifically, how does $F(\epsilon) = |\langle\psi_f(0)|\psi_f(\epsilon)\rangle|^2$ depend on ϵ now? What is the scaling factor n in $(1-F) \propto \epsilon^n$?