Problem Set 4

Due **Tuesday 4/22/2025**

Note: You are not allowed to use AI assistance for generating solutions or code for these problem sets.

If you are using a python notebook, please submit your python notebook ipynb file with your PSet. Please also submit proof that your code runs properly and yields the desired answers. Finally, please clearly mark what sections of your code are for what problems.

Not following these instructions will result in points deducted for this and all future PSets.

Problem 4-1 [5 points] Warm up: Double quantum dot, exchange interaction.
Show that the exchange interaction J that splits the singlet spin state $S_{(1,1)}$ from the triplet states $T^+_{(1,1)}$, $T^0_{(1,1)}$, $T^-_{(1,1)}$ results in the same Hamiltonian as $H = J\vec{S}_1\,\vec{S}_2$, up to a global phase (a global phase takes the form of cI, for a constant c and the identity operator I).

Problem 4-2 [20 points] Resonantly driven CNOT gate for electron spins.
Here we study a cutting-edge experiment using silicon quantum dots that have much longer coherence times than GaAs: D. M. Zajac et al., "Resonantly driven CNOT gate for electron spins," *Science* **359**, 439-442 (2018).

(a) Let's start with Figure 1 in the paper. What is the procedure for initializing the system into $|{\downarrow\downarrow}\rangle$? Illustrate your explanation by sketching a pair of plots showing the important gate voltages V_L and V_R vs. time during the initialization sequence. Then do the same for the measurement sequence. To get reasonable timescales, it may help to look at Fig. S3.

(b) Fig. 2 shows single qubit X rotations on both qubits. These are driven by applying microwaves to the gate S at frequencies around 18 GHz, at the Larmor frequency of the spins.

  (i) The gate S is coupled to both dots, so how are the authors able to separately drive the left and right spins?

  (ii) The authors claim that it is the electric field from the S gate that drives the qubit rotations. Why does this work (section 3 of supplement may be helpful)? What parameter(s) determine the ratio of the electric field E to the spin rotation (Rabi) frequency Ω?

(c) In Fig. 3, the authors show that they can perform rotations on one qubit conditioned on the state of the other. Let's understand why that is the case and reason why the level diagram in 3A is the correct description:

  (i) In class we learned that in the (1,1) charge configuration, the exchange interaction between the dots shifts the singlet state, $(|{\uparrow\downarrow}\rangle - |{\downarrow\uparrow}\rangle)/\sqrt{2}$, by the exchange energy $J = 4t_c^2/E_c$ downwards relative to the triplet states $\{|{\uparrow\uparrow}\rangle, |{\uparrow\downarrow}\rangle + |{\downarrow\uparrow}\rangle)/\sqrt{2}, |{\downarrow\downarrow}\rangle\}$. Plot the energies of the four states as a function of the magnetic field B_z, ranging from $B_z = 0$ to $B_z \gg J$.

(ii) The eigenstates in Fig. 3A are not the same as these, because we have not included the magnetic field gradient. Suppose the field gradient is perfectly antisymmetric, in the sense that the fields on the two dots are $B_L = B_{ext} + \delta B$ and $B_R = B_{ext} - \delta B$, where B_{ext} is the same B_z for both dots, $2\delta B$ is the difference between the dots from the gradient. Show that the addition of δB does not affect the $|\downarrow\downarrow\rangle$ and $|\uparrow\uparrow\rangle$ states. However, it does affect the other two spin states. Write down a 2×2 Hamiltonian for the other two states, including J and δB, and find the eigenstates in the limit $\delta B \gg J$. Now draw all four energy levels. Does it look like Fig. 3A?

(iii) Confirm your result by using QuTip to evaluate the spectrum of the Hamiltonian in the supplementary information S1. Don't include any time dependence.

(d) Once you understand Fig. 3, Fig. 4 is pretty straightforward. However, there is something a little subtle about turning on V_M to implement a two-qubit gate, which is that it pushes the electrons around in the field gradient and does some unwanted z-rotations. It turns out that z rotations acting independently on the two qubits (in other words a rotation on qubit 1 that does not depend on the state of qubit 2) can be compensated later by additional single-qubit gates. However, rotations that are not independent effectively constitute another two-qubit gate and cannot be compensated with additional single qubit rotations. In Fig. S8, the authors characterize this effect. How does this measurement work? In the text, they claim that t = 204 ns is the optimal point. How can this be seen from the plots? Can you speculate how this could be optimized?

Problem 4-3 [18 points] NV center with nuclear spin register.
In this problem, we will look at the NV center and its neighboring nuclear spins as a spin register in the paper G. Waldherr et al., "Quantum error correction in a solid-state hybrid spin register," *Nature* **506**, 204–207 (2014). We will investigate the first part of the paper in this PSet and continue in next week's Pset.

(a) In an ESR spectrum, like the one shown in Fig. 1(b), the transitions of the electron spin from $m_S = 0$ to $m_S = \pm 1$ are measured. These transitions depend on the state of the nuclear register. Please label the different transitions in Fig. 1(b) with the corresponding nuclear register state.

(b) In class we said that the electron spin of the NV center cannot be read out in a single shot at room temperature. Here the authors do however read out the nuclear spins via the electron spin. Explain qualitatively how this is done.

(c) In Fig. 1(d) the authors display a method to perform a gate between the nuclear spins. Why do they call this a non-local gate? Why are they doing this gate via the electron spin, and not performing a two-qubit gate with only nuclear spin operations? Describe how this gate works and explain what is the purpose of the 2π rotation on the electron.

(d) The initialization fidelity of the electron spin at room temperatures is far from ideal and typically around 70%. Nevertheless, the authors demonstrate initialization fidelities of $> 97\%$ for the nuclear register. How is this achieved?

(e) Show that the gate sequence in Fig. 2 (d) indeed results in a W state (you can use QuTiP for this).

(f) Plot the ideal W state density matrix (you can again use QuTiP for this).