Problem Set 3

Due **Tuesday 4/15/2025**

For this homework we define the fidelity between a pure state $|\psi\rangle$ and a general density matrix ρ as

$$F = \langle\psi|\,\rho\,|\psi\rangle\,.$$

This differs from the fidelity function in Qutip, but we use this for simplicity.

Problem 3-1 [30 points] Let's do some error correction!
In this problem, we store one qubit of information, but the physical system suffers from errors. Suppose we model the dynamics of the system with the Lindblad master equation:

$$\frac{d\rho}{dt} = -i\left(H\rho - \rho H\right) + \sum_j \kappa_j \left[J_j\,\rho\,J_j - \frac{1}{2}\left(J_j^\dagger J_j\,\rho + \rho\,J_j^\dagger J_j\right)\right].$$

(a) Suppose we have only one qubit initialized in $|0\rangle$. Ideally we want to keep it unchanged, so $\hat{H} = I$. However, the system may suffer from stochastic X errors, which can be modeled with a master equation by choosing $J = \sigma_x$. We set the error rate $\kappa = 1$. Use QuTiP to simulate the dynamics of the qubit and plot the fidelity $F(t) = \langle 0|\,\rho(t)\,|0\rangle$ as the qubit evolves.

(b) Now suppose we have three qubits and we initialize them in $|000\rangle$. Each of the qubits suffers from stochastic X errors independently, $J_k = \sigma_{x,k}$, at the same error rate $\kappa_k = 1$ (for $k = 1, 2, 3$). Use QuTiP to simulate the dynamics of the 3-qubit system and plot the fidelity $F(t) = \langle 000|\,\rho(t)\,|000\rangle$ as the qubit evolves.

(c) After an evolution time t, we would like to perform a "stabilizer measurement" by measuring Z_1Z_2 and Z_2Z_3. This allows us to decide in which of the four subspaces the state is: $\mathcal{V}_0$ spanned by $\{|000\rangle, |111\rangle\}$, $\mathcal{V}_1$ spanned by $\{|100\rangle, |011\rangle\}$, $\mathcal{V}_2$ spanned by $\{|010\rangle, |101\rangle\}$, or $\mathcal{V}_3$ spanned by $\{|001\rangle, |110\rangle\}$. If you find that the state ends up in $\mathcal{V}_k$ instead of $\mathcal{V}_0$, you can perform a $\hat{\sigma}_{x,k}$ operation on k-th qubit to return it to $\mathcal{V}_0$. This correction operation can be represented as a quantum channel $\mathcal{E}$ such that:

$$\rho_R(t) = \mathcal{E}[\rho(t)] = \sum_{j=0}^{3} R_j \rho(t) R_j^\dagger,$$

where $R_0 = |000\rangle\langle 000| + |111\rangle\langle 111|$ and $R_j = R_0\sigma_{x,j}$ for $j = 1, 2, 3$. Plot the fidelity $F(t) = \langle 000|\,\rho_R(t)\,|000\rangle$ after recovery and compare to the result in (a) and (b). Is it better or worse for different t? Can you briefly explain the reason?

(d) Now we want to perform gate operations. Suppose first we only have one qubit initialized in $|+X\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$. We want to do a Z gate with the Hamiltonian $H = \sigma_z$. If there is no noise, after what time Δt will the state reach $|-X\rangle = (|0\rangle - |1\rangle)/\sqrt{2}$? If during the evolution, there is a stochastic X error with error rate κ, how does the final fidelity $F = \langle -X|\,\rho_f\,|-X\rangle$ change with κ?

(e) We now encode one logical qubit in 3 physical qubits in the following way: $|0_L\rangle = |000\rangle$ and $|1_L\rangle = |111\rangle$. We initialize the logical qubit in $|(+X)_L\rangle = (|0_L\rangle + |1_L\rangle)/\sqrt{2}$, and use Hamiltonian $\hat{H} = (\hat{\sigma}_{z,1} + \hat{\sigma}_{z,2} + \hat{\sigma}_{z,3})/3$ that lasts for time Δt to drive the state toward $|(-X)_L\rangle = (|0_L\rangle - |1_L\rangle)/\sqrt{2}$. However, as in (b), each physical qubit suffers from independent X errors at the same rate κ. In this case, how does the final fidelity $F = \langle(-X)_L|\,\rho_f\,|(-X)_L\rangle$ change with κ (make a plot in QuTiP)? How does it compare to the results in (d)? How does this change if we perform the recovery channel described in (c) for the quantum state ρ_f we get in the end?

(f) (**Bonus! - 3 extra pts**) Suppose for the 3-qubit case, we split the gate time Δt into M intervals, each with equal length (say $M = 100$). At the end of each interval $t_k = k\frac{\Delta t}{M}$, you quickly perform the stabilizer measurement to check whether an X error happened within $t \in [t_{k-1}, t_k]$ and on which qubit the error occurred. You may run into trouble when two or more qubits have X errors in this interval: In such a case, it can for instance happen that X_1X_2 will be misidentified as an X_3 error, and $X_1X_2X_3$ will be identified as no error. Furthermore, you cannot know the exact time an error occurred, so if you find an error in $[t_{k-1}, t_k]$, you may decide that it happened at $t = (t_{k-1} + t_k)/2$. Now that you **approximately know the time and place** of each X_i error during the gate operation, can you design better recovery strategies on the final state in order to achieve higher final fidelity? You may use the idea from quantum trajectories and use the `mcsolve()` function in Qutip to find out for each trajectory when `(Result.col_times)` and where `(Result.col_which)` error happens exactly, but you should assume that you do not have full knowledge of them in practice. You then apply your recovery strategy (supposing these operations are perfect) on each trajectory, and average all the outcomes to get the final fidelity. Discuss the improvement by making use of your protocol.

Problem 3-2 [25 points] Long coherence quantum dots in silicon.
For this problem we will have a closer look at quantum dots realized in silicon, based on the UNSW Dzurek group's publication Veldhorst et al., "An addressable quantum dot qubit with faul-tolerant control-fidelity," *Nature Nanotech.* **9**, 981–985 (2014). We will explore the device design and operation, and will look into gate fidelities and randomized benchmarking (Figure 4 in the paper) at a later stage in the class.

The device is based on a MOS (metal-oxide-semiconductor) silicon structure. Here, the electrons sit at the interface of silicon and silicon dioxide (which naturally forms on top of silicon). Unlike the 2DEGs that we discussed in class, without any gate voltage applied, there are no electrons at the interface. However, when a positive gate voltage is applied, electrons accumulate below the gate and form a 2DEG. You can find more background on MOS silicon structures in F. A. Zwanenburg et al., *Rev. Mod. Phys.* **85**, 961-1019 (2013), especially on p. 980, and also in ref. 19 of Veldhorst et al. above (it is not needed to solve the questions below).

(a) Let's learn about the device (Fig 1a in Veldhorst et al.):

   (i) Specify the signs of the voltages applied to the gates R, G1, G2, G3, G4 and C. Why do these define a dot?

   (ii) Which gate sets the chemical potential of the dot?

(iii) Sketch a cross-section through the device. Indicate the gates, magnetic fields, electrostatic potentials, and the location of the electrons.

(iv) What sets the tunneling rate from the dot to the reservoir?

(v) Estimate the value of B_1 at the dot based on the Rabi frequency that the authors measure in Fig. 2b. What current I_{ESR} does this correspond to? Does this match P_{ESR} reported in the paper?

(vi) Assume that the dot has a vertical confinement of 20nm. Estimate how many nuclear spins the electron spin of the dot interacts with. How does this compare to dots defined in GaAs?

(b) Initialization:

(i) Given the energy scales set by the temperature and by the magnetic field, what is the probability to find a single, completely isolated electron spin in the up state?

(ii) How does this compare to the initialization fidelity of the dot's spin in the paper? What could be the reasons for the difference (if there is a difference)?

(c) Readout:

(i) Figure 1d and Supplementary Figure 1 characterize the single-shot readout of the qubit. Qualitatively sketch the pulses that you would apply to G4 for the "load", "readout" and "empty" phases.

(ii) Based on Supplementary Figure 1, what would be a good "read level" for the readout?

(d) Rabi oscillations:

(i) Take parameters from the paper to simulate Rabi oscillations as a function of detuning using QuTiP (Figure 2d). Can you reproduce the behavior shown in Figure 2d?

(ii) How would that behavior change if you assume a 10 times faster dephasing rate than the authors measure? (please illustrate by simulation in QuTiP)

(e) Coherence times and extending coherence times (Fig. 3):

(i) In Figure 3a the authors perform a Ramsey experiment (similar to Pset 2 problem 2) to determine the dephasing time T_2^*. A slight detuning between the microwave source and the spin causes the qubit to pick up a phase in the rotating frame of the source. Verify this behavior by performing a Ramsey experiment in QuTiP with the microwave on resonance and slightly detuned from resonance. What sets the oscillation frequency?

(ii) Now add dephasing to your Ramsey experiment. Can you reproduce Figure 3a?

(iii) The coherence time of a qubit is not necessarily limited by T_2^* but can often be extended by an echo sequence. Describe briefly and only qualitatively what the difference between Figure 3b and 3c is and why the coherence time in c is longer.