MENG 31500 2026: MIDTERM EXAM

Wednesday Feb 4, 5.30pm - 7.30pm

First Name:	
Last Name:	

Points (for use by graders only):

Q1	
Q2	
Q3	
Q4	

General instructions:

- The exam is closed book, meaning that you are not allowed to use any book, notes, or online resources.
- Calculators, computers, phones, etc. are not needed and not allowed during the exam.
- **Write your answers directly on the exam sheet below the question. If you need more space, use the back of the page, or one of the empty pages at the end of the exam.**
- You should not communicate with anyone during the exam except a TA or faculty member.
- If you have questions during the exam, raise your hand and the TA or faculty member will come to your desk. We will only help clarify the wording of questions, and not provide any other assistance (in order to be fair to all students).
- Make sure you provide some explanation as to how you got your answer. This will allow us to give you partial points even if the final result is incorrect.
- You do NOT have to answer the questions in order! I would suggest first doing all the questions that seem easy to you, and where you know exactly how to proceed.
- Good luck!

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1. *Angular momentum addition [14 points].* Consider two particles where we only consider each particle's angular momentum degrees of freedom. The first particle has angular momentum $S > 0$ (i.e. $j_1 = S$), the second particle has angular momentum $1/2$ (i.e. $j_2 = 1/2$). We want to describe this system using simultaneous eigenstates of the total angular momentum squared (i.e. $\hat{J}^2_{tot}$, quantum number J), and total z angular momentum (i.e. $\hat{J}_{tot,z}$, quantum number M).

   (a) What are the possible values of J and M for our two particle system?

   (b) Let J_{max} be the maximum possible value of J for our two particle system, and M_{max} be the maximum value of M. Consider the state of this system that has the maximum value of $J = J_{max}$ and M_{max}. Express this eigenstate in terms of the product state basis (i.e. states with definite j_1, m_1 and j_2, m_2). Justify your answer.

(c) Now consider the state that has $J = J_{max}$ and $M = M_{max} - 1$. Express this state in terms of the product state basis. Explain each step you use in obtaining your answer.

(d) For the state you found in part (c), if we measure the first particle's z angular momentum, what are the possible outcomes of our measurement? What are the associated probabilities?

(e) Consider the entanglement between particles 1 and 2 in the state (c). Does the entanglement of the state increase or decrease as we increase S? You don't need to explicitly calculate an entanglement measure, but give a quantitative argument supporting your answer.

(f) Now consider the state where $J = J_{max} - 1$, and M has the largest possible value compatible with this value of J. Express this state in the product-state basis. Explain your steps.

2. *Spherical tensors [16 points]*. In what follows, we will use $\hat{T}_k^{(q)}$ to denote spherical tensor operators of rank k. The Wigner-Eckart theorem is the statement:

$$\langle j', m'; \alpha' | \hat{T}_k^{(q)} | j, m; \alpha \rangle = \langle j', m' | k, q; j, m \rangle \langle j'; \alpha' || \hat{T}_k || j; \alpha \rangle \tag{1}$$

   (a) What is the meaning of each factor on the right-hand side of the Wigner-Eckart equation? Describe each factor in 2-3 sentences at most.

(b) Consider the angular momentum operator $\vec{\hat{J}}$. As discussed in class, the angular momentum operators also define a set of tensor operators with $k = 1$. Suppose we are told the value of the following matrix element:

$$\langle j, j | j, j; 1, 0 \rangle = \lambda \tag{2}$$

Use this information and what you know about angular momentum operators to calculate the reduced (or “double bar") matrix element:

$$\langle j'; \alpha' || \hat{J}_1 || j; \alpha \rangle \tag{3}$$

as a function of j, j', α, α' and λ.

(c) Suppose $\vec{\hat{B}}$ is a vector operator. Consider the operator $\hat{Q} = \vec{\hat{J}} \cdot \vec{\hat{B}}$ (where $\vec{\hat{J}}$ is angular momentum). By explicitly deriving how $\hat{Q}$ transforms under rotation , show that it is a $k = 0$ tensor operator. (Don't just write that this is obvious!).

(d) Consider a hydrodgen atom in a weak electric field $\mathscr{E}_z$ in the z direction; we write the Hamiltonian as $\hat{H} = \hat{H}_0 + \hat{W}$, where $\hat{W} = e\mathscr{E}_z\hat{z}$ describes the electric field. Suppose we want to know how this perturbation splits the $n = 3$ energy levels of the hydrogen atom. This involves calculating matrix elements of the perturbation in the degenerate subspace. How many entries in this matrix will be non-zero? How many distinct reduced ("double-bar") matrix elements do we need to calculate? Explain your answer. You do not need to calculate any matrix elements explicitly.

3. *Non-degenerate perturbation theory [8 points].* Consider a system described by a Hamiltonian $\hat{H} = \hat{H}_0 + \hat{W}$. We assume the eigenstates of $\hat{H}_0$ are known, and that $\hat{W}$ is small enough to be treatable using perturbation theory.

   (a) Assume first that $\hat{H}_0$ has no degeneracies, and use an integer n to label its eigenstates (with $n = 1$ being the ground state). Starting from the time-independent Schrodinger equation, derive the general expression for the first order shift in the ground state energy.

(b) Now derive a general expression for the first order shift in the ground state wavefunction (again starting from the time-independent Schrodinger equation)

4. *Degenerate perturbation theory [8 points]*. Consider a system of two harmonic oscillators (lowering operators $\hat{a}$ and $\hat{b}$) with a Hamiltonian $\hat{H} = \hat{H}_0 + \hat{W}$ where:

$$\hat{H}_0 = \omega\left(\hat{a}^\dagger\hat{a} + \hat{b}^\dagger\hat{b}\right) \quad \hat{W} = g\left(\hat{a}^\dagger\hat{b} + \hat{b}^\dagger\hat{a}\right)$$

(a) What are the lowest three energy eigenvalues of $\hat{H}_0$? What is the degeneracy of each of these levels? Call these energies E_1, E_2 and E_3 (with $E_3 > E_2 > E_1$).

(b) Use degenerate perturbation theory to calculate the first order energy shifts to the E_2 energy levels of $\hat{H}_0$. Show all of your steps. You do not need to derive the equations of degenerate perturbation theory.

Extra blank pages if needed

If using these pages to finish answering a question, indicate clearly which question is being addressed.

MENG 315 2026: Midterm Formula Sheet

Angular momentum ladder operators

$$J_\pm = J_x \pm iJ_y$$

$$[J_z, J_\pm] = \pm\hbar J_\pm, \qquad [J_+, J_-] = 2\hbar J_z$$

$$J_\pm |j,m\rangle = \hbar\sqrt{j(j+1) - m(m\pm 1)}\,|j,m\pm 1\rangle$$

Harmonic oscillator ladder operators

$$a = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i\hat{p}}{m\omega}\right), \qquad a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i\hat{p}}{m\omega}\right)$$

$$[a, a^\dagger] = 1$$

$$a|n\rangle = \sqrt{n}\,|n-1\rangle, \qquad a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle$$

Spherical tensor operators

Commutation relations with angular momentum:

$$\boxed{[J_z, T_k^{(q)}] = \hbar q\, T_k^{(q)}}$$

$$\boxed{[J_\pm, T_k^{(q)}] = \hbar\sqrt{k(k+1) - q(q\pm 1)}\, T_k^{(q\pm 1)}}$$

Wigner-Eckart Theorem:

$$\langle j', m'; \alpha' | \hat{T}_k^{(q)} | j, m; \alpha\rangle = \langle j', m' | k, q; j, m\rangle \langle j'; \alpha' || \hat{T}_k || j; \alpha\rangle \tag{1}$$