MENG 31500 2026: FINAL EXAM

Tuesday March 10, 10am - 12pm

First Name:	
Last Name:	

Points (for use by graders only):

Q1	
Q2	
Q3	
Q4	

General instructions:

- The exam is closed book, meaning that you are not allowed to use any book, notes, or online resources.
- Calculators, computers, phones, etc. are not needed and not allowed during the exam.
- **Write your answers directly on the exam sheet below the question. If you need more space, use the back of the page, or one of the empty pages at the end of the exam.**
- You should not communicate with anyone during the exam except a TA or faculty member.
- If you have questions during the exam, raise your hand and the TA or faculty member will come to your desk. We will only help clarify the wording of questions, and not provide any other assistance (in order to be fair to all students).
- Make sure you provide some explanation as to how you got your answer. This will allow us to give you partial points even if the final result is incorrect.
- You do NOT have to answer the questions in order! I would suggest first doing all the questions that seem easy to you, and where you know exactly how to proceed.
- Good luck!

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1. *Stuff from the first half of the course [8 points].*

   (a) $\hat{T}_k^{(q)}$ denotes a set of spherical tensor operators of rank k. The Wigner-Eckart theorem is the statement:

$$\langle j', m'; \alpha' | \hat{T}_k^{(q)} | j, m; \alpha \rangle = \langle j', m' | k, q; j, m \rangle \langle j'; \alpha' \| \hat{T}_k \| j; \alpha \rangle \quad (1)$$

   What is the meaning of each factor on the right-hand side of the Wigner-Eckart equation? Describe each factor in 2-3 sentences at most.

(b) Consider a system of two harmonic oscillators (lowering operators $\hat{a}$ and $\hat{b}$) with a Hamiltonian $\hat{H} = \hat{H}_0 + \hat{W}$ where:

$$\hat{H}_0 = \left(2\omega\hat{a}^\dagger\hat{a} + \omega\hat{b}^\dagger\hat{b}\right) \quad \hat{W} = g\left(\hat{a}^\dagger\hat{b}\hat{b} + \hat{b}^\dagger\hat{b}^\dagger\hat{a}\right)$$

i. What are the lowest three energy eigenvalues of $\hat{H}_0$? (ignore zero-point energies since our $\hat{H}_0$ omits the $\frac{1}{2}\omega$ term) What is the degeneracy of each of these levels? Call these energies E_1, E_2 and E_3 (with $E_3 > E_2 > E_1$).

ii. Use degenerate perturbation theory to calculate the first order energy shifts to the E_3 energy levels (degenerate manifold) of $\hat{H}_0$. Show all of your steps. You do not need to derive the equations of degenerate perturbation theory.

2. *Time-dependent Hamiltonians [16 points].* Consider a one-dimensional simple harmonic oscillator with a time-dependent quadratic drive. The Hamiltonian is (setting $\hbar = 1$):

$$\hat{H} = \omega_0 \hat{a}^\dagger \hat{a} + \lambda \cos(\nu t) \left(\hat{a}\hat{a} + \hat{a}^\dagger \hat{a}^\dagger \right) \equiv \hat{H}_0 + \lambda \hat{V}(t) \tag{2}$$

where $\hat{a}, \hat{a}^\dagger$ are standard harmonic oscillator lowering and raising operators. The frequencies ω_0 and ν are both greater than zero. In what follows we will treat the last time-dependent term (proportional to λ) as a perturbation.

(a) Derive the interaction picture Hamiltonian $\hat{V}_I(t)$ for this system.

(b) We can write the interaction picture wavefunction in terms of the eigenstates $|n\rangle$ of the unperturbed Hamiltonian as

$$|\psi(t)\rangle_I = \sum_n c_n(t)|n\rangle \tag{3}$$

Derive the differential equation that determines the time-evolution of the coefficients $c_n(t)$. Simplify the RHS of this equation using the explicit form of $\hat{V}_I(t)$ you found in part (a).

(c) Suppose at $t = 0$ the system starts in the ground state of $\hat{H}_0$. Calculate the probability amplitude associated with the $n = 2$ eigenstate, $c_2(t)$, for $t > 0$ and **to first order in λ. You can leave your answer as an integral, but the integrand should only involve parameters appearing in the Hamiltonian (no unknown constants). You do not need to explicitly do the integral.**

(d) Using your answer from (c), can the first-order perturbative expression for the probability amplitude ever exhibit an unbounded growth with time t? If so, when does this happen? What is the physical explanation?

(e) Suppose we modify the setup, so that the time-dependent Hamiltonian is now:

$$\hat{H} = \omega_0 \hat{a}^\dagger \hat{a} + \lambda \left(e^{i\nu t} \hat{a}\hat{a} + e^{-i\nu t} \hat{a}^\dagger \hat{a}^\dagger \right) \equiv \hat{H}_0 + \lambda \hat{V}(t) \tag{4}$$

Show that we can make a transformation to a rotating frame such that the Hamiltonian in the rotating frame is now time-independent. Show all of your steps.

(More space for Q 2e if needed)

3. *Fermi's Golden Rule meets second quantization [19 points].*

(a) Explain why the standard Fermi's golden rule expression for a transition rate is not valid at very short times. (3 sentences or less).

(b) Explain why the standard Fermi's golden rule expression for a transition rate is not valid at very long times. (3 sentences or less)

(c) Consider a funny system where a qubit can make transitions between its ground and excited state by absorbing or emitting a free bosonic particle of mass m in one dimension. The Hamiltonian is given by:

$$\hat{H}(t) = \frac{1}{2}\omega_q \hat{\sigma}_z + \sum_k \varepsilon_k \hat{a}_k^\dagger \hat{a}_k + \lambda \cos(\nu t) \sum_k \left(\hat{a}_k^\dagger \hat{\sigma}_- + \hat{\sigma}_+ \hat{a}_k \right), \tag{5}$$

where $\hat{a}_k, \hat{a}_k^\dagger$ are standard second quantized particle destruction and creation operators for a particle with momentum k, and $\varepsilon_k = k^2/2m$. We will treat the last term (proportional to λ) as a perturbation. We will treat the free bosonic particles in the standard way: we assume that they live in a one-dimensional box of length L with periodic boundary conditions, so that the allowed values of k are discrete and given by $k_n = 2\pi n/L$ (with n an integer).

Suppose at $t = 0$ the qubit is in its excited state and there are no particles present. Calculate the Fermi's Golden rule rate(s) for transitions from this initial state to a final state where the qubit is in its ground state and there is one particle present with a momentum k_m (for some specific non-zero choice of m). You do not need to derive Fermi's Golden rule. Your answer can contain an energy conserving delta function if appropriate.

(d) Let's now calculate the **total** transition rate where we sum over all possible momenta k_m of the particle in the final state. Write this total transition rate in terms of an appropriate density of states. Define this density of states in terms of parameters appearing in the Hamiltonian. The density of states can be left as a summation (no integration is needed). Assume that $\omega_q > \nu$.

(e) Explain how your answer in (c) would change if we now consider the case $\omega_q < \nu$. Give a one-sentence intuitive explanation for the difference.

(f) Let's return to the situation in (c), but now consider a different initial state where the qubit is excited, and there are already particles present. Suppose there are M_1 particles in the momentum state k_1, and M_2 particles in the state k_2. All other momentum states have no particles in them. Write this initial state in terms of the vacuum state and second quantized creation operators. Include the correct normalization factor.

(g) For the same Hamiltonian listed in (c) and the initial state given in (f), calculate the Fermi's Golden Rule transition rates to a final state that has exactly M_1 particles in the momentum state k_1, and $M_2 + 1$ particles in the momentum state k_2 (and the qubit in its ground state). Be careful with normalization when calculating matrix elements.

4. *Light matter interaction [8 points].*

(a) A hydrogen atom in its ground state is placed in a weak monochromatic electromagnetic field (frequency ω, linear polarization). Explain why no matter what we pick for the frequency ω of the radiation, to first order in perturbation theory it is impossible for the field to cause a transition to the $2s$ excited state. Assume that we can use the electric dipole approximation. Explain where the relevant selection rules come from, and how you are using symmetry to obtain the result (don't just write the selection rule without explanation).

(b) As discussed in class, when an electron is exposed to an electromagnetic field, the perturbation to the Hamiltonian from the field involves the electron's momentum operator $\vec{\hat{p}}$. However, in calculating Fermi's Golden rule rates, we related things to matrix elements of the electron's position operator, e.g. $\langle f|\hat{x}|i\rangle$. Explain how we obtained this result, and what approximations (if any) were involved.

Extra blank pages if needed

If using these pages to finish answering a question, indicate clearly which question is being addressed.