# Problem Set 1

Due **Tuesday 4/1/2025 in class**

**Problem 1-1 [10 points]** Pauli vector identity and unitary rotation decomposition.

(a) Show that $e^{-i\frac{\theta}{2}\vec{n}\cdot\vec{\sigma}} = \mathbb{I}\cos\left(\frac{\theta}{2}\right) - i\,\vec{n}\cdot\vec{\sigma}\sin\left(\frac{\theta}{2}\right)$, with $\vec{n}$ being a normalized vector. Hint: $\sigma^2_{x,y,z} = \mathbb{I}$.

(b) It is convenient to be able to decompose operations into discrete pulses about orthogonal axes. Show that you can realize an arbitrary unitary in the form:

$$\hat{U} = e^{i\alpha} R_z(\beta) R_y(\gamma) R_z(\delta),$$

where $\alpha, \beta, \gamma, \delta$ are numbers and $R_i(\theta) = e^{-i\frac{\theta}{2}\sigma_i}$.

**Problem 1-2 [10 points]** $\sqrt{SWAP}$ entanglement generation.

The matrix representation of the SWAP gate, $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ at a glance appears similar to that of the CNOT gate, $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$, in the standard two-qubit basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$.
However, unlike the CNOT gate, the SWAP gate cannot generate entanglement. There are several qubit systems in which you can apply a SWAP operation for a variable time. This allows you to perform a "half-SWAP," also called a $\sqrt{SWAP}$, with $\sqrt{SWAP}\sqrt{SWAP} = SWAP$. The $\sqrt{SWAP}$ *is* an entangling gate!

(a) Derive the $4 \times 4$ matrix that represents $\sqrt{SWAP}$.

(b) Show that the $\sqrt{SWAP}$ gate can generate entanglement by creating the $|\phi^+\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$ Bell state from the two-qubit input state $|00\rangle$. (Hint: you may need to use the $\sqrt{SWAP}$ gate more than once along with several single qubit operations)

**Problem 1-3 [20 points]** Spin review and intro to QuTiP
Please submit, **via Canvas**, a Jupyter notebook for this problem along with your pen/paper work.

The Hamiltonian for a spin-1/2 in a magnetic field, $\boldsymbol{B}$, is given by

$$H = -\vec{\mu}\cdot\vec{B} = -\frac{1}{2} g\mu_B \vec{\sigma}\cdot\vec{B},$$

where $g$ is the Landé g-factor, $\mu_B$ is the Bohr magneton, and $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ is a vector of Pauli spin operators.

(a) Write down the matrix and outer product (ket-bra) representations of the three Pauli spin operators and the identity operator.

(b) Find the expectation value of the spin projection, $\langle\vec{\sigma}\rangle = (\langle\sigma_x\rangle, \langle\sigma_y\rangle, \langle\sigma_z\rangle)$, for each of the following quantum states in (1) Dirac braket notation, (2) matrix notation, and (3) on QuTiP:

   (i) $|\psi\rangle = |0\rangle$

   (ii) $|\psi\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$

   (iii) $|\psi\rangle = (\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{2}(1+i)|1\rangle)$

(c) Plot the three states above on the Bloch sphere using QuTiP.

(d) Evaluate the spin Hamiltonian in matrix form assuming the magnetic field points along the $x$ direction, $\vec{B} = (B_x, 0, 0)$. Find its eigenvalues and the corresponding eigenvectors.

(e) Now define the spin Hamiltonian in QuTiP and use it to find the eigenvalues and eigenvectors. For this, use natural units (set $h = 1$ such that your Hamiltonian takes on units of frequency (not radial frequency!) instead of energy), let $g = 2$, $\mu_B = 1.4$ MHz/gauss, and $B_x = 200$ gauss.

(f) Using QuTiP, plot the spectrum (i.e. the eigenenergies) as a function of the magnetic field strength $B_x$. Make sure that your axes are labeled and have units.