MENG 31500 - 2026: Problem Set 6

Due Monday March 2 at 11:59pm (upload PDF via Canvas); late assignments will not be graded

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1. *Qubit thermalization from a bosonic bath (FGR with a mixed initial state).* (For this problem, set $\hbar = 1$ throughout.) Consider a two-level system ("qubit") with

$$\hat{H}_S = \frac{\omega_0}{2}\,\hat{\sigma}_z, \tag{1}$$

coupled weakly to a bosonic bath with

$$\hat{H}_B = \sum_k \omega_k\,\hat{b}_k^\dagger \hat{b}_k, \tag{2}$$

via the interaction

$$\hat{V} = \hat{\sigma}_x\,\hat{B}, \qquad \hat{B} \equiv \sum_k g_k(\hat{b}_k + \hat{b}_k^\dagger), \tag{3}$$

where g_k are (small) coupling constants. Here, each $\hat{b}_k$ operator is a harmonic oscillator lowering operator, with $[\hat{b}_k, \hat{b}_{k'}] = 0$ and $[\hat{b}_k, \hat{b}_{k'}^\dagger] = \delta_{k,k'}$. Assume the total Hamiltonian is $\hat{H} = \hat{H}_S + \hat{H}_B + \hat{V}$ and that Fermi's Golden Rule applies. You may use $\hat{\sigma}_x = \hat{\sigma}_+ + \hat{\sigma}_-$ and the interaction picture defined with respect to $\hat{H}_0 = \hat{H}_S + \hat{H}_B$.

(a) Write the interaction-picture operator $\hat{V}_I(t)$ explicitly in terms of $\hat{\sigma}_\pm$ and $\hat{b}_k, \hat{b}_k^\dagger$. Identify which terms correspond to (i) qubit relaxation $|e\rangle \to |g\rangle$ accompanied by *emission* into the bath, and (ii) qubit excitation $|g\rangle \to |e\rangle$ accompanied by *absorption* from the bath. (You can already anticipate that only energy-conserving terms contribute at long times.)

(b) Now assume the bath is initially in a *pure* number state $|\{n_k\}\rangle \equiv \bigotimes_k |n_k\rangle$. This is a tensor product of energy eigenstates for each bath oscillator, where the bath oscillator k has exactly n_k quanta. Using Fermi's Golden Rule, compute the transition rates

$$\Gamma_\downarrow(\{n_k\}): \quad |e\rangle \otimes |\{n_k\}\rangle \;\to\; |g\rangle \otimes |\{n_k'\}\rangle, \tag{4}$$

$$\Gamma_\uparrow(\{n_k\}): \quad |g\rangle \otimes |\{n_k\}\rangle \;\to\; |e\rangle \otimes |\{n_k''\}\rangle, \tag{5}$$

and show that they involve the expected bosonic "stimulated" factors: for a given mode k resonant with ω_0, emission processes scale as $(n_k + 1)$ while absorption processes scale as n_k.

(c) Now take the bath to be in *thermal equilibrium* at temperature T, i.e.

$$\hat{\rho}_B = \frac{e^{-\beta \hat{H}_B}}{Z}, \qquad \beta \equiv 1/T. \tag{6}$$

To connect to part (b), explicitly write $\hat{\rho}_B$ as a classical mixture of number states:

$$\hat{\rho}_B = \sum_{\{n_k\}} p(\{n_k\})\,|\{n_k\}\rangle\langle\{n_k\}|, \tag{7}$$

and use this to argue that the *physical* FGR rates are obtained by averaging the pure-state rates:

$$\Gamma_\downarrow = \sum_{\{n_k\}} p(\{n_k\})\,\Gamma_\downarrow(\{n_k\}), \qquad \Gamma_\uparrow = \sum_{\{n_k\}} p(\{n_k\})\,\Gamma_\uparrow(\{n_k\}). \tag{8}$$

Show that the result can be written in the continuum limit as

$$\Gamma_\downarrow = 2\pi\,J(\omega_0)\,\big[n_B(\omega_0) + 1\big], \qquad \Gamma_\uparrow = 2\pi\,J(\omega_0)\,n_B(\omega_0), \tag{9}$$

where

$$J(\omega) \equiv \sum_k |g_k|^2\,\delta(\omega - \omega_k), \qquad n_B(\omega) \equiv \frac{1}{e^{\beta\omega} - 1}. \tag{10}$$

(Helpful intermediate step: because $\hat{H}_B = \sum_k \omega_k \hat{b}_k^\dagger \hat{b}_k$ is a sum of independent modes, $\hat{\rho}_B$ factorizes over k, and $\langle n_k\rangle_{\text{th}} = n_B(\omega_k)$.)

(d) Use the rates from part (c) to write a rate equation for the excited-state population $p_e(t)$:

$$\frac{dp_e}{dt} = -\Gamma_\downarrow p_e + \Gamma_\uparrow (1 - p_e). \tag{11}$$

Solve for $p_e(t)$ given $p_e(0)$, identify the equilibration rate (i.e. an effective T_1^{-1}), and show that the stationary solution satisfies

$$\frac{\Gamma_\uparrow}{\Gamma_\downarrow} = e^{-\beta\omega_0}, \qquad \Rightarrow \qquad p_e(\infty) = \frac{1}{1 + e^{\beta\omega_0}}. \tag{12}$$

Briefly explain (one sentence) how this differs from the "classical noise" case discussed in lecture, where the noise spectrum is symmetric and one would instead have $\Gamma_\uparrow = \Gamma_\downarrow$.

2. *TRK sum rule.* Consider a particle of mass M described by a Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2M} + V(\hat{x}), \tag{13}$$

where V is a function only of position. Let $\hat{H}|m\rangle = E_m|m\rangle$. Define the oscillator strength

$$f_{mi} = \frac{2M(E_m - E_i)}{\hbar^2} \left|\langle m|\hat{x}|i\rangle\right|^2. \tag{14}$$

The TRK sum rule is

$$\sum_m f_{mi} = 1. \tag{15}$$

(a) Calculate the double commutator $[\hat{x}, [\hat{H}, \hat{x}]]$, showing that it is proportional to the unit operator.

(b) Use this result to prove the TRK sum rule. Hint: start with the summation on the LHS of the TRK sum rule, expand out the absolute value squared, make things look like the double commutator in part (a).