MENG 31500 - 2026: Problem Set 5

Due Monday Feb. 23 at 11:59pm (upload PDF via Canvas); late assignments will not be graded

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1. *Generalized Rabi oscillations.* (Set $\hbar = 1$.) Let's generalize the discussion of Rabi oscillations from class to the case of a spin-1 system (i.e. a three level quantum system). Consider a spin-1 particle subject to a static magnetic field along $\hat{z}$ and an oscillating magnetic field in the xy plane. The time-dependent Hamiltonian is

$$H(t) = \omega_0 \hat{S}_z + \frac{\Omega_1}{2}\left(e^{-i\omega t}\hat{S}_+ + e^{i\omega t}\hat{S}_-\right), \tag{1}$$

where ω_0 is the energy splitting between adjacent m-levels, ω is the drive frequency, and Ω_1 is the amplitude of the driving field. $\hat{S}_z, \hat{S}_\pm$ are standard spin-1 operators.

You may use the standard spin-1 matrix representations, where the raising and lowering operators satisfy

$$S_\pm |m\rangle = \sqrt{s(s+1) - m(m \pm 1)}\; |m \pm 1\rangle, \qquad s = 1,$$

so that, in the ordered basis $\{|{+1}\rangle, |0\rangle, |{-1}\rangle\}$,

$$S_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad S_+ = \sqrt{2}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad S_- = \sqrt{2}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

(a) Make a rotating frame transformation so that in the new frame, the transformed wavefunction obeys a time-independent Hamiltonian. Explicitly write out the time-dependent unitary, and derive the time-dependent Schrödinger equation in the new frame. Write the effective Hamiltonian in the new frame as a 3×3 matrix in the ordered basis $\{|{+1}\rangle, |0\rangle, |{-1}\rangle\}$.

(b) Let $\delta \equiv \omega_0 - \omega$ be the detuning of the drive frequency from resonance. Show that the rotating-frame Hamiltonian has a zero eigenvalue. What is the corresponding eigenvector? Call this state $|D\rangle$.

(c) What does the state $|D\rangle$ look like in the original lab-frame (i.e. Schrodinger picture)? How do the probabilities associated with different $\hat{S}_z$ eigenstates evolve in time in this state?

(d) Give an intuitive explanation for the surprising behavior for the probabilities.

(e) Using the fact that you already know about the zero-eigenvalue eigenvector, find the remaining two energy eigenvalues and eigenvectors of the rotating-frame Hamiltonian. Suppose in the original lab frame (i.e. the Schrodinger picture) we start in some arbitrary state, and measure some arbitrary observable $\hat{O}$. What are the possible oscillation frequencies we will see in $\langle \hat{O}(t) \rangle$? (up to the trivial ω-precession from the rotating-frame transformation)

(f) Consider now the case of resonant driving, $\delta = 0$. If the system starts in the state $|\psi(0)\rangle = |{-1}\rangle$, find the lab-frame state $|\psi(t)\rangle$ and compute the probabilities $P_m(t) = \left|\langle m|\psi(t)\rangle\right|^2$ for $m = +1, 0, -1$. Plot these probabilities as a function of time.

(g) Again at resonance, repeat part (f) for the initial state $|\psi(0)\rangle = |0\rangle$. Plot these probabilities as a function of time. Comment on the qualitative differences from part (f).

2. *Sudden perturbation to a Harmonic oscillator.* Suppose we have a 1D simple harmonic oscillator, described by the Hamiltonian

$$\hat{H}_0 = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}k\hat{x}^2 \tag{2}$$

where $\hat{x}, \hat{p}$ are the usual position and momentum operators. The full Hamiltonian is given by

$$\hat{H}(t) = \hat{H}_0 + \theta(t)\,\frac{\Delta k}{2}\,\hat{x}^2 \tag{3}$$

The second term is a time-dependent perturbation where the spring constant of the oscillator is suddenly changed from k to $k + \Delta k$ ($\theta(t)$ is the Heaviside step function). We assume that at $t = 0$ the system is prepared in the ground state of the harmonic oscillator

(a) Write the Schrodinger equation in the interaction picture for this system. Write an explicit form for the interaction-picture perturbation operator $\hat{V}_I(t)$ in terms of harmonic oscillator raising and lowering operators.

(b) Using first order time-dependent perturbation theory, what is the probability $p_n(t)$ for finding the system at $t > 0$ in the number state with n quanta (for all $n \geq 0$)?

(c) Show that for $t > 0$, you can diagonalize the Hamiltonian $\hat{H}(t)$ (as it is time-independent). What are the energy eigenvalues?

(d) Using this result, show that the exact result for $p_n(t)$ must be a periodic function. How can this be consistent with your result in part (b)?