MENG 31500 - 2026: Problem Set 4

Due Monday Feb. 16 at 11:59pm (upload PDF via Canvas); late assignments will not be graded

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1. *Wavefunction corrections from Schrieffer-Wolff transformations.* The standard problem in perturbation theory is a Hamiltonian $\hat{H} = \hat{H}_0 + \lambda \hat{W}$. In class we discussed how SW transformations can be used to derive an approximate "simple" Hamiltonian $\hat{H}'$ that captures the effects of the perturbation $\hat{W}$ to order λ^2. We worked out a concrete example of a collection of N qubits coupled to a cavity where the cavity and qubits had a large energy detuning. Our focus in class was understanding how energies change to second order in λ. Here we will ask how wavefunctions change.

   (a) Consider the approximate transformed Hamiltonian $\hat{H}'$ for the qubits-plus-cavity system given on p8 of the lecture 8b+9 notes. Show that this Hamiltonian commutes both with total photon number and $\hat{J}_z$. We can thus characterize the energy eigenstates with two quantum numbers n (photon number) and m (J_z value).

   (b) What is the energy eigenvalue associated with a state with a photon number n and a $\hat{J}_z$ value m?

   (c) We understand what the eigenstates of $\hat{H}'$ are, but we really want the eigenstates of the original Hamiltonian $\hat{H}$ (e.g. as written on p4 of the lecture notes). Show first that $\hat{H}$ and $\hat{H}'$ have the same energy eigenvalues to order λ^2.

   (d) Let $|n, m\rangle$ be an eigenstate of $\hat{H}'$ with eigenvalue E. Find the corresponding eigenstate of $\hat{H}$ that has the same energy eigenvalue. We want this eigenstate to first order in λ. Write this state as a superposition of eigenstates of $\hat{H}_0$. Do this calculation using the SW transformation and not standard first order perturbation theory. (Hint: use the defining relation between H and H' written at the top of p3).

   (e) Show your answer in part (d) matches what you would obtain using standard first order perturbation theory for a wavefunction correction.

   (f) Consider the case where $\omega_c \gg \Omega$, and an unperturbed eigenstate of $\hat{H}_0$ that has $N_e < N$ of the qubits in the excited state, and zero photons in the cavity. Consider now the first order in λ correction to this state (as found in part d). Using only this first-order state, what is the probability that the state has a non-zero photon population?

   (g) Use the calculation above to argue that even if $g << |\omega_c - \Omega|$, perturbation theory will break down if N is large enough.

2. *Using SW transformations to understand a two-qubit gate.* Consider a system of two qubits (1 and 2) that have very different splitting frequencies, where a drive is applied to qubit 1, but at a frequency matching the splitting frequency of qubit 2. The two qubits only weakly interact with one another, with an interaction that moves excitations between the two qubits. In an appropriate rotating frame (to be discussed in Lec. 11) the Hamiltonian is:

$$\hat{H} = \frac{\Delta_1}{2}\hat{\sigma}_z^{(1)} + \frac{\Omega}{2}\hat{\sigma}_x^{(1)} + \lambda g \left(\hat{\sigma}_+^{(2)}\hat{\sigma}_-^{(1)} + \hat{\sigma}_+^{(1)}\hat{\sigma}_-^{(2)}\right) \quad (1)$$

We use Pauli matrices to describe each qubit (exactly like in class). Here Δ_1 represents the energy difference (detuning) between the two qubits, Ω is the amplitude of the drive on qubit 1, and λg is the interaction between them. We will take Δ_1 to be the largest frequency in $\hat{H}$, and treat the interaction λg as a perturbation in what follows.

(a) Suppose $\Omega = 0$ (no drive). Explain intuitively why (in the regime of interest $\Delta_1 \gg g$) the interaction between the two qubits will be extremely weak. What sort of interactions would we get to order λ^2? How would the rate scale with g and Δ_1? Don't do any explicit calculations, instead justify your answer using symmetry. (Hint: take inspiration from the qubits-cavity setup discussed in lecture).

(b) We want to make a SW transformation to eliminate the coupling between the two qubits to first order. The transformation is generated by a unitary $\hat{U} = e^{-i\lambda\hat{S}}$, and our goal is to find $\hat{S}$ to order λ^0. Do this for the case $\Omega = 0$.

(c) We now want to use the $\hat{S}$ you found to get the transformed Hamiltonian $\hat{H}' = e^{i\lambda\hat{S}}\hat{H}_0 e^{-i\lambda\hat{S}}$. Let's now include the Ω drive term in doing this calculation in $\hat{H}_0$ (but not in $\hat{S}$!). Find $\hat{H}'$ to order λ^1.

(d) Explain in physical terms the meaning of each term in $\hat{H}'$.

(e) Explain how this interaction could be used to generate entanglement between two qubits that are initially in a product state.

(f) It might seem strange to have dropped Ω when calculating $\hat{S}$ and then put it back again when calculating $\hat{H}'$. When can this be justified? To make things more precise, we can replace $\Omega \to \lambda_2\Omega$, where λ_2 is another small parameter like λ.