MENG 31500 2026: Problem Set 3

Due Monday Feb 2 (by midnight) via Canvas upload. Late assignments will not be graded

1. *Generalizing EPR style arguments to three particle entangled states.* Consider three spin 1/2 particles (A,B,C) described by the state vector

$$|\psi\rangle = \frac{1}{2}\left(|+++\rangle - |+--\rangle - |-+-\rangle - |--+\rangle\right)$$

where each term is a product state of definite $\hat{S}_z$ for each particle (i.e. $|+-+>$ is state where A has $S_z = +1/2$,B has $S_z = -1/2$ and C has $S_z = +1/2$). Each particle is sent to a different observer (Alice, Bob and Carl). Each measures either the z or x component of the spin of their particle, and records the value. As with the EPR setup, we can assume their measurements are spacelike separated, and thus cannot influence one another. After repeating the experiment many times (each time with a new three-particle system prepared in the state $|\psi\rangle$), Alice, Bob and Carl get together and examine their results for each run of the experiment.

   (a) If each observer measures spin along z, what are the possible outcomes of the three measurements? Suppose we focus on just two of the observers, with them both measuring S_z. Are there correlations in their measurements? Instead, suppose we look at all three observers (again, with everyone measuring along z)? Are there correlations between all three measurements? In both cases, support your answer by looking at an appropriate expectation value.

   (b) Suppose Alice and Bob now measure their spin along x, while Carl still measures his spin along z. Show that the only possible outcomes of the measurements correspond to an even number of $(+1/2)$ outcomes being recorded.

   (c) Suppose next that Alice measures along x, while Bob and Carl measure along z. Show that in this case, the results of the measurements is completely random.

   (d) Argue from symmetry that the results from (b) also apply any time only one of the three observers measures along z (with the other two measuring along x). The same argument implies that the results from (c) applies any time only one of the observers measures along x.

   (e) Let's now construct an EPR-like argument. Consider runs where Bob and Carl measure along z. Show that in such runs, if Bob and Carl get together and compare their results, they can predict what Alice would have measured in the run if she also measured along z. Argue (like EPR) that this implies particle A must have a predetermined value of S_z. Be sure to explain your reasoning.

   (f) Similarly, suppose Bob measures along z and Carl along x. Show that in such runs, if Bob and Carl get together and compare their results, they can predict what Alice would get if she had measured along x. Argue (like EPR) that this implies particle A must have a predetermined value of S_x. Again, be sure to explain your reasoning.

   (g) Combining (e) and (f), we see that EPR-type reasoning implies particle A must have a definite value of both S_z and S_x. Using part (d), the same argument applies to particles B and C. Thus, EPR reasoning implies each particle has a definite S_z and S_x. We could thus try to construct a "hidden variable" theory like what we did in the EPR case. Show that it is impossible to construct such a theory. (Hint: try to write down a table of all the kinds of states having definite properties, each labelled by the value of a hidden variable λ, that might be consistent with the properties of $|\psi\rangle$. Then argue that no matter how you pick probabilities for the hidden variable λ you cannot capture the statistics of the state $|\psi\rangle$).

2. *Superdense coding and security*. Consider the superdense coding protocol described in lectures. Recall that the last step involves Alice sending one qubit to Bob; the final two-qubit state that Bob ends up with perfectly encodes the 2-bit message Alice wanted to send. Suppose that in this last step, a malicious evesdropper (named Eve by convention) intercepts Alice's qubit. Argue rigorously that no matter what Eve measures on this intercepted qubit, she learns nothing (even in a probabilistic sense) about the message Alice was trying to send.

3. *Perturbative treatment of a light-matter interaction.* Consider a two level atom (modelled as a qubit or spin 1/2) interacting with a single mode of an optical or microwave cavity; this single mode is described as a harmonic oscillator. The simplest interaction we could imagine is one where excitations can move from the atom to the cavity, and vice versa. This is described by the Hamiltonian:

$$\hat{H} = \omega_{cav}\hat{a}^\dagger\hat{a} + \frac{\omega_{at}}{2}\hat{\sigma}_z + g\left(\hat{a}^\dagger\hat{\sigma}_- + h.c.\right)$$

Here $\hat{\sigma}_z$is a Pauli matrix (i.e. the S_z operator for a spin 1/2, divided by 1/2); $\hat{\sigma}_- = |-\rangle\langle+|$ is the corresponding lowering operator. In what follows, we will treat the last term $\propto g$ as a perturbation, and assume $\omega_{cav} > \omega_{at}$.

(a) What is the ground state of this Hamiltonian when $g = 0$? Explain why this remains an eigenstate for any value of g.

(b) Consider the first excited state of H_0 (i.e H when $g = 0$). Using perturbation theory, calculate the first order energy shift of this state.

(c) Calculate the first order wavefunction correction of the state in (b) using perturbation theory.

(d) Calculate the second order energy shift of the state in (b) using perturbation theory.

(e) Show that $\hat{H}$conserves the total number of excitations (photons + atomic excitations). Use this to calculate the first excited state of $\hat{H}$ exactly for all values of g. Confirm that the energy you get matches the results of perturbation theory to order g^2.

4. *Perturbing a harmonic oscillator.* Consider a simple model of a harmonic oscillator in 1D (position operator $\hat{x}$, momentum operator $\hat{p}$) where we add a small anharmonic term to the potential that is quartic in position. This turns out to be a reasonable model for a superconducting transmon qubit (where momentum and position are replaced by charge and phase variables). The anharmonic oscillator is described by the Hamiltonian (setting hbar to 1):

$$\hat{H} = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{x}^2 + \lambda\omega\left(\frac{\hat{x}}{x_{zpt}}\right)^4$$

Here $x_{zpt} = \sqrt{1/2m\omega}$ and $\lambda << 1$ is a small dimensionless constant. Our approach to this Hamiltonian will be to treat the last term as a perturbation.

(a) Calculate the first order energy shift of the nth energy eigenstate.

(b) Calculate the second oder energy shift of the nth energy eigenstate. (Hint: in the notes, see how did the example of a perturbation that was a cubic potential).

(c) Is the second order energy shift in this system negative for all states? Try to explain heuristically why this is (or is not) the case.

(d) Make an estimate for the largest value of n where we expect the perturbative treatment to be valid.