MENG 31500 2026: Problem Set 2

Due Monday Jan 26 (by midnight) via Canvas upload. Late assignments will not be graded

1. *Spherical tensor operator basics***.** In what follows, $\hat{T}^k_q$ denotes spherical tensors using the same notation used in class and the textbooks.

   (a) Using the commutation relation of a spherical tensor with $\hat{J}_z$, derive the selection rule that constrains m' in the matrix element $\langle j', m'|\hat{T}^k_q|j,m\rangle$ (i.e. for what values of m' is this matrix element non-zero?).

   (b) Show that if $\hat{V}_j$ form a vector operator, then $\hat{V}_z, \mp(\hat{V}_x \pm i\hat{V}_y)/\sqrt{2}$ are tensor operators with $k=1$. Do this by checking the commutation of these operators with $\hat{J}_z$ and $\hat{J}_\pm$

   (c) We argued that commuting angular momentum operators with a spherical tensor is analogous to acting with an angular momentum operator on an eigenket $|j,m\rangle$. The one case we did not consider is the J^2 operator. To this end, compute the commutator $\sum_i [\hat{J}_i, [\hat{J}_i, \hat{T}^k_q]]$ using the commutation relations that were presented in class. Show that this double commutator is proportional to $\hat{T}^k_q$. What is the constant of proportionality? In what sense does this remind us of the action of $\hat{J}^2$ on a state $|j,m\rangle$?

2. *Rotations and spherical tensors.*

   (a) Consider a particle with total angular momentum $j=1$. Let R denote a rotation by an angle ϕ around the x axis. The unitary operator corresponding to this rotation is $\hat{D}[R]$, and the matrix elements of this operator between $|j=1,m\rangle$ states forms a 3x3 matrix $D^1_{q',q}$. Calculate this matrix, showing your steps.

   (b) Consider a vector operator $\hat{V}_j$. Use it to construct spherical tensor operators $\hat{T}^1_q$. Use the matrix $D^1_{q',q}$ to calculate how these spherical tensor operators transform under the rotation R introduced in part (a).

   (c) Show that the expressions obtained in (b) match the expected transformation properties of $\hat{V}_x, \hat{V}_y, \hat{V}_z$.

3. *Spherical tensors as vectors in an abstract space*. Spherical tensors are defined by the fact that these operators transform under rotations like angular momentum eigenkets. Their commutation relations with angular momentum operators also remind us of how angular momentum operators act on eigenkets. In this problem, we will make this connection a bit more precise by viewing spherical tensors as vectors in an abstract space. Note that the space of all operators acting on a Hilbert space is itself a vector space (i.e. taking linear combinations of operators gives us a new allowed operator, etc.).

   (a) Let's consider ordinary operators to be akin to vectors. We then need a notation of operators on this space, i.e. linear operations that map ordinary operators to operators. We call such objects super-operators (i.e. linear operations that map ordinary operators to new operators). We define the rotation superoperator $\check{D}[R]$ by the rule that $\check{D}[R]$ maps an input operator $\hat{O}$ to the rotated version of the operator, i.e.

   $$\check{D}[R].\hat{O} \equiv \hat{D}[R]\hat{O}\hat{D}^{-1}[R]$$

   So for every possible rotation R we have a corresponding super-operator $\check{D}[R]$. Show that these form a good representation of the rotation group (this is easy).

   (b) We can now consider an infinitesimal rotation (where the rotation angle ϕ goes to zero). We can then use this to define the angular momentum superoperator via

   $$\check{D}[R] \to \check{1} - i\phi \sum_{j=x,y,z} n_j \check{J}_j$$

   Here $\check{1}$ is the identity super-operator: for any operator $\hat{O}$, $\check{1}.\hat{O} = \hat{O}$. Use this to find the definition of the J superoperators (i.e. give an equation that specifies how each of these superoperators maps an input operator to a new operator).

(c) Re-write the commutation relations that define a spherical tensor in terms of the angular momentum superoperators you found above. Show that the form of the equations now completely mimics the action of angular momentum operators on angular momentum eigenkets.

4. *Two spin ensembles*. In quantum information and AMO physics, we are often interested in setups where we have groups of qubits that act collectively like a single large angular momentum (i.e. the individual spin 1/2's add together to give the maximum possible total angular momentum). Suppose we have two ensembles like this (i.e. two large spins), with angular momentum operators $\hat{J}_j^{(1)}$for ensemble 1 $(j=x,y,z)$,and $\hat{J}_j^{(2)}$for ensemble 2. As usual, we can form a total angular momentum operator $\hat{J}_j^{tot}=\hat{J}_j^{(1)}+\hat{J}_j^{(2)}$ and use this total angular momentum operator to define the notion of a spherical tensor. Spherical tensors behave "nicely" under joint rotations of both ensembles. The following operators play an interesting role when we make the two spin ensembles interact with one another.

   (a) First, a preliminary result. Prove that if the operators $\hat{A}_q^k$ $(q=-k,...,k)$ and $\hat{B}_q^k$ $(q=-k,...,k)$ are spherical tensors of rank k, then the operators $\hat{C}_q^k=\alpha\hat{A}_q^k+\beta\hat{B}_q^k$ are also spherical tensors of rank k.

   (b) Consider the operator $\hat{D}_z=\hat{J}_z^{(1)}-\hat{J}_z^{(2)}$ (i.e. the difference in the z angular momentum between ensemble 1 and ensemble 2). Show that this operator is in fact a spherical tensor. What are the values of k and q ?

   (c) Consider the operator $\hat{D}_{int}=\hat{J}_+^{(1)}\hat{J}_-^{(2)}-\hat{J}_+^{(2)}\hat{J}_-^{(1)}$. Shows that this operator is also a spherical tensor. What are the corresponding values of k and q? (Hint: you can use Eq. 3.10.27 in Sakurai that tells you how to take products of spherical tensors to construct new spherical tensors).

   (d) Suppose one has a Hamiltonian $\hat{H}=\lambda_z\hat{D}_z+\lambda_{int}\hat{D}_{int}$, and take each $\hat{J}_j^{(\alpha)}(\alpha=1,2)$ to represent the total angular momentum of $N/2$ spin 1/2's (i.e. qubits). Give a physical interpretation to each term in this Hamiltonian. Do you expect the ground state of this Hamiltonian to have entanglement between ensemble 1 and 2? Explain your reasoning (you do not have to calculate anything).

   (e) Let $|J,m\rangle$be total angular momentum eigenkets, i.e. joint eigenkets of $\hat{J}_z^{tot}$ and $(\hat{J}^{tot})^2$. Suppose we want to know how our Hamiltonian can cause transitions from a total angular momentum state $|J,m\rangle$ to another total angular momentum state $|J+1,m\rangle$(i.e. a state with the z total angular momentum, but one more unit of total angular momentum). We would then need matrix elements of the $\hat{D}^\alpha$ operators. Consider matrix elements defined as $F^\alpha(J,m)=\langle J+1,m|\hat{D}_\alpha|J,m\rangle$ with $\alpha=z,int$. Consider the ratio $F^{int}(J,m)/F^z(J,m)$. Show that this ratio is independent of m.