MENG 31500 2026: Problem Set 1

Due Monday Jan 19 (by midnight) via Canvas upload. Late assignments will not be graded

1. *Two spin 1/2's, basics.* Consider two spin 1/2's, where for each spin individually, the J_z eigenstates are denoted $|+\rangle, |-\rangle$ (same notation as class). Suppose the two spins are describe by the normalizd quantum state

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|+-\rangle + i|-+\rangle\right)$$

   (a) Write $|\psi\rangle$ using basis states $|J,M\rangle$ of definite total angular momentum

   (b) If one measures the magnitude-squared of the total angular momentum, i.e. the operator $\hat{J}^2_{tot}$, in this state, what are the possible outcomes? What are the probabilities associated with these outcomes?

   (c) Compute the expectation value of $\vec{J}_1 \cdot \vec{J}_2$ in the state $|\psi\rangle$.

   (d) Compute the reduced density matrix of the first spin (spin 1) in this state. What is the von Neumann entropy $S_{vn} \equiv -Tr(\hat{\rho} log_2 \hat{\rho})$ of this state? How big is this entropy compared to its maximum possible value?

   (e) Suppose we make a measurement of $\hat{J}^2_{tot}$ in this state and get a result 2. What is the post-measurement state? If we then make a measurement of J^z_{tot}, what are the possible outcomes and associated probabilities?

2. *Constructing CG Coefficients.* Consider two angular momenta 1 and 2. The first particle has an angular momentum quantum number $j_1 = 1$, the second particle has an angular momentum quantum number $j_2 = 1$

   (a) Consier a basis of total angular momentum states for this system. What are the possible values of J and M? Show that the number of total angular momentum basis states is the same as the number of basis states in the product basis.

   (b) Compute the CG coefficients for this system using the method discussed in class (i.e. explicitly construct the $|J,M\rangle$ states in terms of the product state basis kets).

   (c) Which of these total angular momentum states have the largest entanglement? How large is the entanglement of these states compared to the maximum possible value for an arbitrary state in the Hilbert space? You can use the von Neumann entropy of the reduced density matrix as a metric of entanglement.

   (d) Which of these states are even under exchange of spins 1 and 2? Which are odd? Verify that this matches the general statement on the exchange symmetry of CG coefficients discussed in class and Shankar Chapter 15.

3. *Multiple spins.* In cavity QED, we often deal with the systems with multiple two level atoms that all couple to a common photonic mode of a cavity. Each individual atom can be modeled as a spin 1/2 (with ground and excited states corresponding to J_z eigenstates $|\pm\rangle$. As the cavity interacts to all the spins collectively, it is useful to describe this system in terms of states of total angular momentum. The total angular momentum is defined in the usual way: for N spins, $\vec{J}_{tot} = \sum_{k=1}^{N} \vec{J}_k$. For simplicity, lets consider the case where we have only 3 two-level atoms in the cavity (i.e. 3 effective spin 1/2's).

   (a) The tensor product space desribing the three spins is denoted $(1/2) \otimes (1/2) \otimes (1/2)$ (i.e. the tensor product of three spin 1/2 Hilbert spaces). Suppose instead we want to describe the system using states of definite total angular momentum. Based on expectation for classical addition of vectors, what will the possible values of the total angular momentum quantum number j_{tot} be? How many possible states does this naively give us?

   (b) The number of basis states you find in (a) will be smaller than the number of product state basis states. The only resolution is that there is degeneracy among the states of total angular momentum: there are different orthogonal states that have the same quantum number J, M. Explain intuitively why this is, and which $|J,M\rangle$ states will be degenerate.

   (c) Explicitly find the expressions for the $|J,M\rangle$ states in terms of the product-basis eigenkets using the method discussed in class. Hint: first "add" spins 1 and 2, once you've done this add spins 2 and 3.

(d) Consider now the spins plus cavity system. The cavity photons are modeled as a harmonic oscillator with a lowering operator $\hat{a}$. Suppose the Hamiltonian describing the spins plus cavity system is given by

$$\hat{H} = \omega_c \hat{a}^\dagger \hat{a} + \omega_{at} \hat{J}^z_{tot} + g\left(\hat{a}^\dagger \hat{J}^-_{tot} + \hat{J}^+_{tot} \hat{a}\right)$$

(e) Show that this Hamiltonian conserves $\hat{J}^2_{tot}$ but does not conserve $\hat{J}^z_{tot.}$ by computing relevant commutators.

(f) Show this Hamiltonian conserves the total number of excitations $\hat{Q} = \hat{a}^\dagger \hat{a} + J^{\hat{z}}_{tot}$.

(g) Find the energy eigenstates of the system with 0 and 1 total excitations. Write the corresponding eigenkets in terms of states with total angular momentum.